

Four ways across the wall

Boris Pioline

Laboratoire de Physique Théorique et Hautes Energies,
CNRS UMR 7589 and Université Pierre et Marie Curie - Paris 6,
4 place Jussieu, 75252 Paris cedex 05, France

E-mail: pioline@lpthe.jussieu.fr

Abstract. An important question in the study of $\mathcal{N} = 2$ supersymmetric string or field theories is to compute the jump of the BPS spectrum across walls of marginal stability in the space of parameters or vacua. I survey four apparently different answers for this problem, two of which are based on the mathematics of generalized Donaldson-Thomas invariants (the Kontsevich-Soibelman and the Joyce-Song formulae), while the other two are based on the physics of multi-centered black hole solutions (the Coulomb branch and the Higgs branch formulae, discovered in joint work with Jan Manschot and Ashoke Sen [1]). Explicit computations indicate that these formulae are equivalent, though a combinatorial proof is currently lacking.

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1. Introduction

In quantum field theories and string theory vacua with extended supersymmetry, it is often possible to determine the spectrum of BPS bound states in some weakly coupled region of moduli (or parameter) space \mathcal{B} . In extrapolating the BPS spectrum to strong coupling, one usually faces two issues: i) short BPS multiplets may pair up into long multiplets and leave the BPS spectrum and ii) single-particle bound states may decay into the continuum of multi-particle states. The first issue can be avoided by considering a suitable index $\Omega(\gamma, t)$, designed such that contributions from long multiplets cancel. $\Omega(\gamma, t)$ is then a piecewise constant function of the charge vector γ and couplings/moduli $t \in \mathcal{B}$.

The second issue arises at certain loci in moduli space, where the bound state becomes unstable towards decay into a n -particle state with charges $\{\alpha_i\}$ such that $\gamma = \sum_{i=1\dots n} \alpha_i$. In four-dimensional field or string theories with $\mathcal{N} = 2$ supersymmetry, the mass of a BPS bound state $M(\gamma, t)$ is equal to $|Z(\gamma, t)|$, where the central charge Z is a map from \mathcal{B} to $\text{Hom}(\Gamma, \mathbb{C})$, where Γ is the charge lattice. In particular, Z is linear in its first argument γ . The decay is therefore energetically possible only when (and even then, marginally so) the phase of $Z(\gamma, t)$ aligns with the phase of each of the $Z(\alpha_i, t)$'s, so that $M(\gamma) = \sum_{i=1\dots n} M(\alpha_i)$. This alignment takes place in a locus of codimension $p - 1$ in the moduli space \mathcal{B} , where p is the dimension of the subspace of Γ spanned by the α_i 's. The dangerous case is $p = 2$, where the locus defines a codimension one “wall of marginal stability” in \mathcal{B} , across which the index $\Omega(\gamma, t)$ may jump. A paradigm of this phenomenon is Seiberg-Witten theory with $SU(2)$ gauge group and no flavors: across the curve $\{a/a_D \in \mathbb{R}^+\}$ in the u -plane, the BPS spectrum jumps from an infinite number

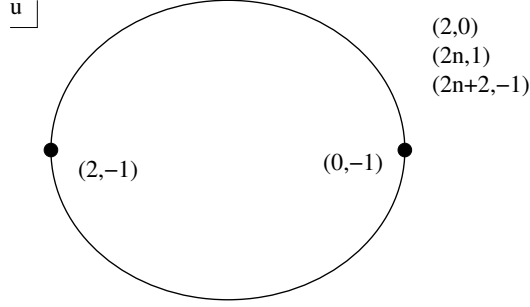


Figure 1. Chamber structure of the u -plane and BPS spectrum in $\mathcal{N} = 2, D = 4$ SYM theory with $SU(2)$ gauge group and no flavor. The line $\text{Im}(a/a_D) = 0$ separates the strong and weak coupling chambers. The only stable BPS states in the strong coupling chamber are the monopole and dyons with charges $(q, p) = \pm(0, 1), \pm(2, -1)$, in the conventions of [5]. The weak coupling spectrum consists of these same states and their images around the monodromy at infinity, plus the W -boson with charge $(2, 0)$.

of states in the weak coupling region to just two states in the strong coupling region – the monopole and the dyon (see Fig. 1) [2, 3]. An important physical question is therefore to determine the jump

$$\Delta\Omega(M\gamma_1 + N\gamma_2) = \Omega^-(M\gamma_1 + N\gamma_2) - \Omega^+(M\gamma_1 + N\gamma_2) \quad (1.1)$$

of the BPS index ¹ $\Omega(\gamma, t) \equiv \text{Tr}'(-1)^F$ across the wall of marginal stability

$$W(\gamma_1, \gamma_2) = \{t \in \mathcal{B} / \arg[Z(\gamma_1, t)] = \arg[Z(\gamma_2, t)]\}, \quad (1.2)$$

in terms of the BPS indices on side of the wall, say the $\Omega^\pm(\gamma)$'s. Here we denoted by $\Omega^\pm(\gamma)$ the index in the chamber c^\pm on the side of the wall where $\arg Z(\gamma_1, t) \gtrless \arg Z(\gamma_2, t)$. It will be convenient to choose the basis γ_1, γ_2 such that $\Omega^+(M\gamma_1 + N\gamma_2)$ vanishes whenever $MN < 0$ (the ‘root basis’ condition [4]), and denote by $\tilde{\Gamma} = (\mathbb{Z}^+\gamma_1 + \mathbb{Z}^+\gamma_2) \setminus \{0\}$ the positive cone in the two-dimensional sublattice spanned by γ_1, γ_2 . We further assume that $\langle \gamma_1, \gamma_2 \rangle < 0$.

As we shall explain in §4.2, in $\mathcal{N} = 2$ supergravity and for suitably large charges, the jump in $\Omega(M\gamma_1 + N\gamma_2, t)$ is accounted by the loss or gain of a family n -centered BPS black hole solutions with charges $\alpha_i = M_i\gamma_1 + N_i\gamma_2 \in \tilde{\Gamma}$, with $\sum_i (M_i, N_i) = (M, N)$, which exist on the side c_- of the wall, and on this side only. Close to the wall, the n -centered configuration is loosely bound, with the relative distances r_{ij} between the centers diverging at the wall. As a result, its index factorizes into the product of the internal index $\Omega(\alpha_i)$ associated to each center, times the index of the degrees of freedom associated to the relative motion of the centers (with suitable modifications due to Bose-Fermi statistics when some of the (M_i, N_i) coincide) [1]. In §4.2, we shall compute this configurational index using localization techniques, and obtain $\Delta\Omega(M\gamma_1 + N\gamma_2)$ for arbitrary values of (M, N) .

A closely similar problem arises in the mathematics of Donaldson-Thomas invariants of coherent sheaves on a compact complex manifold \mathcal{X} . These invariants, which we shall again denote by $\Omega(\gamma, t)$, are labelled by a class γ in the K-theory lattice $\Gamma = K(\mathcal{X})$, and depend on a choice of stability condition $\phi_t : \Gamma \rightarrow S^1$ inside a complex family parametrized by t . $\Omega(\gamma, t)$ is defined as the Euler characteristic ² of the moduli space $\mathcal{M}(\gamma, t)$ of stable coherent sheaves on \mathcal{X}

¹ Here, Tr' denotes the trace in the Hilbert space associated to (γ, t) with the center of motion degrees of freedom removed, and $(-1)^F$ denotes the fermionic parity, equal to $(-1)^{2J_3}$ by the spin-statistics relation, where J_3 is the angular momentum operator along the z axis.

² Rather, the Euler characteristic weighted by Behrend’s function, see [6, 7] for details.

in the class γ with respect to ϕ_t . As the stability condition is varied, some of the stable sheaves may become unstable and the DT invariant $\Omega(\gamma, t)$ may jump. This happens on the same walls of marginal stability as in (1.2), where $\phi_t(\gamma)$ plays the role of $\arg Z(\gamma, t)$. The similarity between these two wall-crossing problems follows from the fact that stable objects in the derived category of coherent sheaves on a Calabi-Yau three-fold \mathcal{X} are realized physically by BPS states in type IIA string theory compactified on \mathcal{X} (see e.g. [8, 9] for reviews).

In two independent pieces of work, Kontsevich-Soibelman [10] and Joyce-Song [7] have determined the variation $\Delta\Omega(\gamma)$ in terms of the DT invariants on one side of the wall. In both works, it was noted that the wall-crossing formula takes a simpler form in terms of the ‘rational DT invariants’ $\bar{\Omega}(\gamma, t)$, related to the ordinary, integer-valued invariants $\Omega(\gamma, t)$ by the ‘multi-cover formula’

$$\bar{\Omega}(\gamma, t) \equiv \sum_{m|\gamma} \Omega(\gamma/m, t)/m^2, \quad (1.3)$$

where the sum runs over all integers $m \geq 1$ such that $\gamma/m \in \Gamma$ (thus, $\bar{\Omega}(\gamma) = \Omega(\gamma)$ if γ is primitive). From a physics point of view, we shall explain in §4.1 that the replacement $\Omega(\gamma, t) \rightarrow \bar{\Omega}(\gamma, t)$ effectively converts the Bose-Fermi statistics of the n centers into Boltzmannian statistics, thereby allowing us to treat the centers as distinguishable. In more detail, the KS and JS formulae express the jump $\Delta\bar{\Omega}(\gamma)$ as

$$\Delta\bar{\Omega}(\gamma, t) = \sum_{n \geq 2} \sum_{\substack{\{\alpha_1, \dots, \alpha_n\} \in \tilde{\Gamma} \\ \gamma = \alpha_1 + \dots + \alpha_n}} \frac{g(\{\alpha_i\})}{\prod m_k!} \prod_{i=1}^n \bar{\Omega}^+(\alpha_i, t), \quad (1.4)$$

where the second sum runs over all (unordered) decompositions of the total charge vector γ into a sum of n vectors $\alpha_i \in \tilde{\Gamma}$. The coefficients $g(\{\alpha_i\})$ (after extracting out a Boltzmann-Gibbs symmetry factor $\prod m_k!$ whenever $\{\alpha_i\}$ contains m_1 copies of β_1 , m_2 copies of β_2 , etc) are universal functions of the α_i ’s. As we shall see, the function $g(\{\alpha_i\})$ turns out to be equal to the index $\text{Tr}'(-1)^F$ of the configurational degrees of freedom of an n -centered Boltzmann black hole solution. To compute this index, it is convenient to consider the refined (or equivariant) index $g(\{\alpha_i\}, y) = \text{Tr}'(-y)^{2J_3}$, evaluate the latter by localization methods, and set $y = 1$ at the end. In fact, the refined configurational index $g(\{\alpha_i\}, y)$ for general y enters the wall-crossing formula for so-called motivic Donaldson-Thomas invariants, see (2.11).

In the remainder of this survey, we first give an executive summary of the KS (§2) and JS (§3) wall-crossing formulae, silencing the subtleties involved in defining the Donaldson-Thomas invariants by themselves. In §4 we then derive the combinatorial factors $g(\{\alpha_i\})$ by quantizing the phase space of multi-centered BPS black holes and evaluating the index by localization. In §5, we give an alternative computation of $g(\{\alpha_i\})$ relying on Reineke’s results for quivers without closed loops. We end in §6 with a discussion of some open problems. The material is mostly based on [1, 11], which the reader should consult for more details. Other important references include [12, 13, 14, 15, 16, 10, 7, 17, 18].

2. The Kontsevich-Soibelman wall-crossing formula

We start by reviewing the Kontsevich-Soibelman wall-crossing formula for generalized Donaldson-Thomas invariants. For generality, we first state its motivic (aka refined) version, and then discuss its classical limit. As an application, we use the KS formula to rederive the primitive and semi-primitive wall-crossing formula. Finally, we extract the coefficient $g(\{\alpha_i\})$ appearing in (1.4) for $n \leq 3$, for comparison with other wall-crossing formulae discussed later.

2.1. The motivic KS formula

The motivic KS formula pertains to ‘motivic Donaldson-Thomas invariants’ $\Omega_n(\gamma, t)$ attached to a Calabi-Yau threefold category \mathcal{X} , with a stability condition ϕ_t . Informally, $\Omega_n(\gamma, t)$ is the

$n + \frac{1}{2}d$ -th Betti number of the moduli space $\mathcal{M}(\gamma, t)$ of stable degree- γ objects in the triangulated category of coherent sheaves on \mathcal{X} , where $d = \dim \mathcal{M}(\gamma, t)$. We define the Poincaré polynomial

$$\Omega_{\text{ref}}(\gamma, t, y) = \sum_{n \in \mathbb{Z}} (-y)^n \Omega_n(\gamma, t), \quad (2.1)$$

which is a finite Laurent polynomial in y , symmetric under $y \rightarrow 1/y$ (the subscript ‘ref’ stands for ‘refined’, which for our purposes is synonymous with ‘motivic’ [17]).

To state the KS formula, we introduce the Lie algebra \mathcal{A} spanned by abstract generators e_γ , for each $\gamma \in K(\mathcal{X}) = \Gamma$, subject to the commutation rule

$$[e_\gamma, e_{\gamma'}] = \kappa(\langle \gamma, \gamma' \rangle, y) e_{\gamma+\gamma'}, \quad (2.2)$$

where $\langle \gamma, \gamma' \rangle$ is the integer-valued antisymmetric pairing on Γ (physically, the Dirac-Schwinger-Zwanziger product for electromagnetic charge vectors), and

$$\kappa(x, y) \equiv \frac{(-y)^x - (-y)^{-x}}{y - 1/y} = (-1)^x \frac{\sinh(\nu x)}{\sinh \nu}, \quad \nu \equiv \ln y. \quad (2.3)$$

It is straightforward to check that (2.2) satisfies the Jacobi identity for any y .

Now, for a given choice of stability condition t and any $\gamma \in \Gamma$, let $U_\gamma(t)$ be the element in the group $\mathcal{G} = \exp(\mathcal{A})$ defined by

$$U_\gamma(t) = \prod_{n \in \mathbb{Z}} \mathbf{E} \left(\frac{y^n e_\gamma}{y - 1/y} \right)^{(-1)^{n+1} \Omega_n(\gamma, t)}, \quad (2.4)$$

where $\mathbf{E}(x)$ is the quantum dilogarithm function

$$\mathbf{E}(x) \equiv \exp \left[\sum_{k=1}^{\infty} \frac{(xy)^k}{k(1 - y^{2k})} \right]. \quad (2.5)$$

We now restrict to $\gamma \in \tilde{\Gamma}$, where $\tilde{\Gamma}$ is the positive cone in the two-dimensional sublattice of Γ spanned by two primitive vectors γ_1, γ_2 , and assume that the ‘root basis’ condition stated below (1.2) holds. The motivic KS wall-crossing formula [10, 17, 18] states that the following two ordered products

$$\prod_{\substack{M \geq 0, N \geq 0, \\ M/N \downarrow}} U_{M\gamma_1 + N\gamma_2}^+ = \prod_{\substack{M \geq 0, N \geq 0, \\ M/N \uparrow}} U_{M\gamma_1 + N\gamma_2}^-, \quad (2.6)$$

where the products are ordered with decreasing (resp., increasing) values of $M/N \in [0, +\infty]$ (such that the argument of $Z(\alpha)$ decreases from left to right on either side). Here, U_γ^\pm denote the group element $U_\gamma(t)$ when t lies on the side c_\pm of the wall $\mathcal{W}(\gamma_1, \gamma_2)$. Thus, assuming that the $\Omega_n^+(\gamma)$ ’s are known for all $\gamma \in \tilde{\Gamma}$, the $\Omega_n^-(\gamma)$ ’s can be computed by re-ordering the product on the l.h.s. of (2.6) in the opposite order, using the commutation rule (2.2) and the Baker-Campbell-Hausdorff (BCH) formula, and reading off the exponents $\Omega_n^-(\gamma)$.

This procedure is vastly simplified by expressing the products in (2.6) in terms of the ‘rational motivic invariants’

$$\bar{\Omega}_{\text{ref}}(\gamma, t, y) \equiv \sum_{m|\gamma} \frac{(y - y^{-1})}{m(y^m - y^{-m})} \Omega_{\text{ref}}(\gamma/m, t, y^m). \quad (2.7)$$

The relation between $\Omega_{\text{ref}}(\gamma, y)$ and $\bar{\Omega}_{\text{ref}}(\gamma, y)$ is easily inverted by means of the Möbius formula,

$$\Omega_{\text{ref}}(\gamma, t, y) = \sum_{m|\gamma} \mu(m) \frac{(y - y^{-1})}{m(y^m - y^{-m})} \bar{\Omega}_{\text{ref}}(\gamma/m, t, y^m), \quad (2.8)$$

where $\mu(d)$ is the Möbius function (1 if d is a product of an even number of distinct primes, -1 if d is a product of an odd number of primes, or 0 otherwise). Using the fact that the generators $e_{\ell\gamma}$ commute for all $\ell \in \mathbb{Z}$, we may rewrite (2.9) as a product of factors labelled by coprime (M, N) ,

$$\prod_{\substack{M \geq 0, N \geq 0 > 0, \\ \gcd(M, N) = 1, M/N \downarrow}} V_{M\gamma_1 + N\gamma_2}^+ = \prod_{\substack{M \geq 0, N \geq 0 > 0, \\ \gcd(M, N) = 1, M/N \uparrow}} V_{M\gamma_1 + N\gamma_2}^-, \quad (2.9)$$

where

$$V_{\gamma}(t) = \prod_{\ell \geq 1} U_{\ell\gamma}(t) = \exp \left(\sum_{N=1}^{\infty} \bar{\Omega}_{\text{ref}}(N\gamma, t, y) e_{N\gamma} \right). \quad (2.10)$$

Due to the fact that the algebra (2.2) is graded by $\tilde{\Gamma}$, and that every factor of e_{α} in the logarithm of V_{γ} is multiplied by a factor of $\Omega_{\text{ref}}(\alpha, y)$ for the same vector α , it is clear that the result of the re-ordering procedure outlined below (2.6) will produce a wall-crossing formula of the form

$$\bar{\Omega}_{\text{ref}}^-(\gamma, y) - \bar{\Omega}_{\text{ref}}^+(\gamma, y) = \sum_{n \geq 2} \sum_{\substack{\{\alpha_1, \dots, \alpha_n\} \in \tilde{\Gamma}^n \\ \gamma = \alpha_1 + \dots + \alpha_n}} \frac{g_{\text{ref}}(\{\alpha_i\}, y)}{\prod m_k!} \prod_{i=1}^n \bar{\Omega}_{\text{ref}}^+(\alpha_i, y), \quad (2.11)$$

where the second sum runs over unordered sets of n charge vectors $\alpha_i = M_i\gamma_1 + N_i\gamma_2$ such that $\sum_{i=1 \dots n} \alpha_i = \gamma = M\gamma_1 + N\gamma_2$. By the 'root basis' property, only a finite number of terms appear on the right-hand side. In §2.2 below, we shall compute the universal combinatorial factors $g_{\text{ref}}(\{\alpha_i\}, y)$ in selected cases.

Before doing so however, we discuss the classical (or numerical) KS wall-crossing formula, which arises from the motivic formula (2.9) in the limit $y \rightarrow 1$. In this limit, the Poincaré polynomial (2.1) reduces to the Euler-Behrnd characteristic of the moduli space $\mathcal{M}(\gamma, t)$

$$\Omega_{\text{ref}}(\gamma, t, y) \xrightarrow{y \rightarrow 1} \chi[\mathcal{M}(\gamma, t)] \in \mathbb{Z}, \quad (2.12)$$

while $\bar{\Omega}_{\text{ref}}(\gamma, y)$ reduces to the 'rational DT invariant'

$$\bar{\Omega}_{\text{ref}}(\gamma, t, y) \xrightarrow{y \rightarrow 1} \bar{\Omega}(\gamma, t) \equiv \sum_{m|\gamma} m^{-2} \Omega(\gamma/m, t). \quad (2.13)$$

Moreover, the commutation rule (2.2) in the Lie algebra \mathcal{A} has a smooth limit

$$[e_{\gamma}, e_{\gamma'}] = (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle e_{\gamma + \gamma'}, \quad (2.14)$$

and so does the operator V_{γ} in (2.10). As a result, the combinatorial coefficients $g_{\text{ref}}(\{\alpha_i\}, y)$ have a smooth limit as $y \rightarrow 1$, and the wall-crossing formula for the rational DT invariants is given by the limit of (2.11) as $y \rightarrow 1$, i.e. Eq. (1.4) with

$$g(\{\alpha_i\}) = \lim_{y \rightarrow 1} g_{\text{ref}}(\{\alpha_i\}, y). \quad (2.15)$$

2.2. Primitive and semi-primitive wall-crossing

Despite the fact that either side of the KS wall-crossing formula (2.9) involves an infinite number of factors, the procedure of re-ordering the product involves only a finite number of operations, for the following reason (already hinted at below (2.11)): for any $M, N \geq 0$,

$$\mathcal{I}_{M,N} \equiv \left\{ \sum_{m>M \text{ and/or } n>N} \mathbb{R} \cdot e_{m\gamma_1+n\gamma_2} \right\} \quad (2.16)$$

is a two-sided ideal in \mathcal{A} , and the quotient $\mathcal{A}_{M,N} = \mathcal{A}/\mathcal{I}_{M,N}$ is a finite dimensional algebra. For the purpose of computing $\Delta\Omega(M\gamma_1 + N\gamma_2)$, it is sufficient to project the relation (2.9) to $\mathcal{A}_{M,N}$ and use the truncation of the BCH formula at order $\min(M, N)$. E.g. to compute $\Delta\Omega(\gamma_1 + \gamma_2)$, it suffices to re-order the l.h.s of the identity in $\mathcal{A}_{1,1}$

$$\begin{aligned} & \exp(\Omega^+(\gamma_1)e_{\gamma_1}) \exp(\Omega^+(\gamma_1 + \gamma_2)e_{\gamma_1+\gamma_2}) \exp(\Omega^+(\gamma_2)e_{\gamma_2}) \\ &= \exp(\Omega^-(\gamma_2)e_{\gamma_2}) \exp(\Omega^-(\gamma_1 + \gamma_2)e_{\gamma_1+\gamma_2}) \exp(\Omega^-(\gamma_1)e_{\gamma_1}) \end{aligned} \quad (2.17)$$

using the truncated BCH formula $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}$, and match the result to the r.h.s. In this way, we find that the motivic invariants $\Omega_{\text{ref}}(\gamma_1, y), \Omega_{\text{ref}}(\gamma_2, y)$ are constant across the wall, while

$$\Delta\Omega_{\text{ref}}(\gamma_1 + \gamma_2, y) = \kappa(\langle \gamma_1, \gamma_2 \rangle, y) \Omega_{\text{ref}}(\gamma_1, y) \Omega_{\text{ref}}(\gamma_2, y) . \quad (2.18)$$

This relation (or its obvious classical limit at $y = 1$) is known as the ‘primitive wall crossing formula’ [14, 19, 17]. From (2.18), setting $\alpha_1 = \gamma_2$ and $\alpha_2 = \gamma_1$ so that $\langle \alpha_1, \alpha_2 \rangle > 0$, we read off the combinatorial factor

$$g_{\text{ref}}(\alpha_1, \alpha_2, y) = -\kappa(\langle \alpha_1, \alpha_2 \rangle, y) = (-1)^{\langle \alpha_1, \alpha_2 \rangle + 1} \frac{\sinh(\nu \langle \alpha_1, \alpha_2 \rangle)}{\sinh \nu} . \quad (2.19)$$

Up to a sign, this is recognized as the character $\text{Tr}(-y)^{2J_3}$ of a representation of $SU(2)$ with spin $j = \frac{1}{2}(|\langle \alpha_1, \alpha_2 \rangle| - 1)$.

With some more work, one can easily extract the combinatorial coefficients $g_{\text{ref}}(\{\alpha_i\}, y)$ for $n > 2$. E.g, for $n = 3$ and $\alpha_1, \alpha_2, \alpha_3$ three distinct (non necessarily primitive) elements of $\tilde{\Gamma}$ ordered such that $\alpha_{ij} \equiv \langle \alpha_i, \alpha_j \rangle > 0$ for $i < j$, we find

$$g_{\text{ref}}(\alpha_1, \alpha_2, \alpha_3, y) = (-1)^{\alpha_{12}+\alpha_{13}+\alpha_{23}} \frac{\sinh(\nu\alpha_{12}) \sinh(\nu(\alpha_{13} + \alpha_{23}))}{\sinh^2 \nu} \quad (2.20)$$

when $\alpha_{12} > \alpha_{23}$, or

$$g_{\text{ref}}(\alpha_1, \alpha_2, \alpha_3, y) = (-1)^{\alpha_{12}+\alpha_{13}+\alpha_{23}} \frac{\sinh(\nu\alpha_{23}) \sinh(\nu(\alpha_{12} + \alpha_{13}))}{\sinh^2 \nu}$$

when $\alpha_{12} < \alpha_{23}$. The result for $n = 4$ can be found in [1].

While the amount of work necessary to extract the combinatorial factors quickly grows with (M, N) , for fixed (small) M it is possible to compute all the jumps for $\gamma \rightarrow M\gamma_1 + N\gamma_2$ at once using the Hadamard lemma $\log(e^X Y e^{-X}) = \sum_{n \geq 0} \text{Ad}_X^n \cdot Y / n!$, where $\text{Ad}_X \cdot Y \equiv [X, Y]$. Defining

$$Z^\pm(M, q, y) = \sum_{N=0}^{\infty} \Omega_{\text{ref}}^\pm(M, N, y) q^N, \quad \Omega_{\text{ref}}^\pm(M, N, y) \equiv \Omega_{\text{ref}}^\pm(M\gamma_1 + N\gamma_2, y), \quad (2.21)$$

we find, for $M = 1$ [20, 1], the ‘semi-primitive’ wall crossing formula [14]

$$Z^-(1, q, y) = Z^+(1, q, y) Z_{\text{halo}}(\gamma_1, q, y) \quad (2.22)$$

where

$$Z_{\text{halo}}(\gamma_1, q, y) \equiv \exp \left(\sum_{\ell=1}^{\infty} \kappa(\ell \langle \gamma_1, \gamma_2 \rangle, y) \bar{\Omega}_{\text{ref}}(\ell \gamma_2, y) q^\ell \right). \quad (2.23)$$

The reason for the subscript ‘halo’ will become apparent in §4.1. Re-expressed in terms of the integer motivic invariants, this can be written as an infinite product [18]

$$Z_{\text{halo}}(\gamma_1, q, y) = \prod_{\substack{k \geq 1, n \in \mathbb{Z} \\ 1 \leq j \leq k|\gamma_{12}|}} \left(1 - (-1)^{k|\gamma_{12}|} q^k y^{n+2j-1-k|\gamma_{12}|} \right)^{(-1)^n \Omega_n(k\gamma_2)} \quad (2.24)$$

Generalizations of (2.22) for $M = 2, 3$ can be found in [1].

2.3. Exact wall crossing

Finally, we discuss some examples where the re-ordering of the product in (2.9) can be performed in the full untruncated algebra \mathcal{A} . Suppose that in the chamber c_+ , the only non-vanishing DT invariants are $\Omega^+(\gamma_1)$ and $\Omega^+(\gamma_2)$. If $\gamma_{12} = -1$, the result of the re-ordering gives³

$$U_{\gamma_2} U_{\gamma_1} = U_{\gamma_1} U_{\gamma_1+\gamma_2} U_{\gamma_2}, \quad \gamma_{12} = -1, \quad (2.25)$$

which follows from the pentagonal identity for the quantum dilogarithm function. If instead $\gamma_{12} = -2$, one arrives at [10]

$$U_{(2,-1)} \cdot U_{(0,1)} = U_{(0,1)} \cdot U_{(2,1)} \cdot U_{(4,1)} \cdots U_{(2,0)} \cdots U_{(3,-1)} \cdot U_{(2,-1)} U_{(1,-1)}, \quad (2.26)$$

where we denoted $\gamma_2 = (0, 1)$, $\gamma_1 = (2, -1)$ to match the usual basis of electromagnetic charges in Seiberg-Witten theory with $G = SU(2)$ and no flavors [5]. As first noted by Denef, Eq. (2.26) then embodies the BPS spectrum of this gauge theory on the two sides of the curve of marginal stability $\text{Im}(a/a_D) = 0$, see Fig. 1. Analogues of (2.26) for $SU(2)$ gauge theories with $0 < N_f < 4$ flavors can be found in [15, 18]. More general identities of this type can be derived using Y-systems and cluster algebra techniques, see e.g. [21, 22, 23].

3. The Joyce-Song wall-crossing formula

In this section, we briefly review the Joyce-Song wall-crossing formula, which was derived in the context of the Abelian category of coherent sheaves on a Calabi-Yau three-fold \mathcal{X} [7]. Unlike the KS formula, the JS formula only applies to the jump of the classical (or numerical) Donaldson-Thomas invariants. Moreover, it gives a fully explicit formula for the combinatorial factors $g(\{\alpha_i\})$ appearing in (1.4). The price to pay is that the JS formula is computationally less efficient, as it involves sums over many terms with large denominators and large cancellations.

To state the JS formula, we first introduce S , U and \mathcal{L} factors, which are functions of an ordered list of charge vectors $\alpha_i \in \tilde{\Gamma}$, $i = 1 \dots n$:

- We define $S(\alpha_1, \dots, \alpha_n) \in \{0, \pm 1\}$ as follows. If $n = 1$, set $S(\alpha_1) = 1$. If $n > 1$ and, for every $i = 1 \dots n-1$, either

$$\begin{aligned} (a) \quad & \langle \alpha_i, \alpha_{i+1} \rangle \leq 0 \quad \text{and} \quad \langle \alpha_1 + \dots + \alpha_i, \alpha_{i+1} + \dots + \alpha_n \rangle < 0, \quad \text{or} \\ (b) \quad & \langle \alpha_i, \alpha_{i+1} \rangle > 0 \quad \text{and} \quad \langle \alpha_1 + \dots + \alpha_i, \alpha_{i+1} + \dots + \alpha_n \rangle \geq 0, \end{aligned} \quad (3.1)$$

let $S(\alpha_1, \dots, \alpha_n) = (-1)^r$, where r is the number of times option (a) is realized; otherwise, $S(\alpha_1, \dots, \alpha_n) = 0$.

³ In (2.25) and (2.26), $\Omega = 1$ for each factor, except for the factor $U_{(2,0)}$ in the middle of (2.26), for which $\Omega = -2$.

- To define the U factor (not to be confused with the operator U of the previous section !), consider all ordered partitions of the n vectors α_i into $1 \leq m \leq n$ packets $\{\alpha_{a_{j-1}+1}, \dots, \alpha_{a_j}\}$, $j = 1 \dots m$, with $0 = a_0 < a_1 < \dots < a_m = n$, such that all vectors within each packet are collinear. Let

$$\beta_j = \alpha_{a_{j-1}+1} + \dots + \alpha_{a_j}, \quad j = 1 \dots m \quad (3.2)$$

be the sum of the charge vectors in each packet. Next, consider all ordered partitions of the m vectors β_j into $1 \leq l \leq m$ packets $\{\beta_{b_{k-1}+1}, \dots, \beta_{b_k}\}$, with $0 = b_0 < b_1 < \dots < b_l = m$, $k = 1 \dots l$, such that the total charge vectors $\delta_k = \beta_{b_{k-1}+1} + \dots + \beta_{b_k}$, $k = 1 \dots l$ are all collinear. Define the U -factor as the sum

$$U(\alpha_1, \dots, \alpha_n) \equiv \sum_l \frac{(-1)^{l-1}}{l} \cdot \prod_{k=1}^l \prod_{j=1}^m \frac{1}{(a_j - a_{j-1})!} S(\beta_{b_{k-1}+1}, \beta_{b_{k-1}+2}, \dots, \beta_{b_k}) . \quad (3.3)$$

over all partitions of α_i and β_j satisfying the conditions above. If none of the α_i are parallel, $S = U$. Contributions with $l > 1$ arise only when $\{\alpha_i\}$ can be split into two (or more) packets with the same total charge, e.g.

$$U[\gamma_1, \gamma_2, \gamma_1, \gamma_2] = S[\gamma_1, \gamma_2, \gamma_1, \gamma_2] - \frac{1}{2} S[\gamma_1, \gamma_2]^2 = 1 - \frac{1}{2} (-1)^2 = \frac{1}{2} \quad (3.4)$$

- Finally, departing from the notations in [7], define the \mathcal{L} factor by

$$\mathcal{L}(\alpha_1, \dots, \alpha_n) = \sum_{\text{trees}} \prod_{\text{edges}(i,j)} \langle \alpha_i, \alpha_j \rangle \quad (3.5)$$

where the sum runs over all labelled trees with n vertices labelled $\{1, \dots, n\}$, with edges oriented from i to j if $i < j$. There are n^{n-2} labelled trees with n -vertices, which can be labelled by their Prüfer code, an arbitrary sequence of $n - 2$ numbers in $\{1, \dots, n\}$.

With these definitions, the result of [7] can be stated as an explicit formula for the combinatorial factors $g(\{\alpha_i\})$ appearing in (1.4):

$$g(\{\alpha_i\}) = \frac{1}{2^{n-1}} (-1)^{n-1+\sum_{i<j} \langle \alpha_i, \alpha_j \rangle} \sum_{\sigma \in \Sigma_n} \mathcal{L}(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}) U(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}) . \quad (3.6)$$

As an illustration, we now use the JS formula to derive the combinatorial factors $g(\{\alpha_i\})$ for $n = 2, 3$. For $n = 2$ (assuming as before that $\gamma_{12} < 0$), the U, S, \mathcal{L} factors are given in the following table

$\sigma(12)$	S	U	\mathcal{L}
12	a	-1	γ_{12}
21	b	1	$-\gamma_{12}$

(3.7)

The JS formula (3.6) then leads to

$$g(\gamma_1, \gamma_2) = (-1)^{\gamma_{12}} \gamma_{12} \Omega(\gamma_1) \Omega(\gamma_2) , \quad (3.8)$$

in agreement with the classical limit of (2.19).

For $n = 3$, assuming as before that $\alpha_1, \alpha_2, \alpha_3$ are three distinct (non necessarily primitive) elements of $\bar{\Gamma}$ ordered such that $\alpha_{ij} \equiv \langle \alpha_i, \alpha_j \rangle > 0$ for $i < j$ and moreover that $\alpha_{12} > \alpha_{23}$, we find that the S, U, \mathcal{L} factors are given by

$\sigma(123)$	S	U	\mathcal{L}
123	bb	1	$\alpha_{12}\alpha_{13} + \alpha_{13}\alpha_{23} + \alpha_{12}\alpha_{23}$
132	b-	0	$\alpha_{12}\alpha_{13} - \alpha_{13}\alpha_{23} - \alpha_{12}\alpha_{23}$
213	ab	-1	$-\alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{23} - \alpha_{12}\alpha_{13}$
231	-a	0	$\alpha_{12}\alpha_{13} - \alpha_{13}\alpha_{23} - \alpha_{12}\alpha_{23}$
312	ab	-1	$\alpha_{13}\alpha_{23} - \alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{12}$
321	aa	1	$\alpha_{13}\alpha_{23} + \alpha_{12}\alpha_{13} + \alpha_{12}\alpha_{23}$

(3.9)

The JS formula (3.6) then leads to

$$g(\{\alpha_1, \alpha_2, \alpha_3\}) = (-1)^{\alpha_{12} + \alpha_{23} + \alpha_{13}} \alpha_{12} (\alpha_{13} + \alpha_{23}) , \quad (3.10)$$

in agreement with the classical limit of (2.20). The computation for $n = 4$ can be found in [1], and matches the result from the KS formula.

4. Wall-crossing from multi-centered quantum black holes

In this section, we give a new physical derivation of the wall-crossing formula (1.4) (in particular, a new formula for the combinatorial factors $g(\{\alpha_i, y\})$) based on the quantum mechanics of multi-centered black hole configuration. Before starting, it should be noted that the KS wall-crossing formula (and to a lesser extent, the JS formula) has already been derived or interpreted in various physical settings [19, 15, 24, 25, 26, 27, 28, 4, 29]. Our derivation is arguably more elementary, as it relies only on the supersymmetric quantum mechanics of point particles interacting by Coulomb and Lorentz-type forces. The down-side is that it does not make the algebra \mathcal{A} manifest and, admittedly, relies on some plausible but not rigorously proven assumptions.

4.1. From Bose-Fermi to Boltzmann statistics

To motivate our approach, let us return to the semi-primitive wall-crossing formula (2.22) and for simplicity, concentrate on the classical limit $y = 1$. Substituting (2.13) in (2.23), one may rewrite (2.22) as

$$\frac{\sum_{N \geq 0} \Omega^-(1, N) q^N}{\sum_{N \geq 0} \Omega^+(1, N) q^N} = \prod_{k > 0} \left(1 - (-1)^{k\gamma_{12}} q^k \right)^{k |\gamma_{12}| \Omega^+(k\gamma_2)} . \quad (4.1)$$

E.g. for $\gamma \mapsto \gamma_1 + 2\gamma_2$, we find

$$\begin{aligned} \Delta\Omega(1, 2) = & 2\gamma_{12} \Omega^+(1, 0) \Omega^+(0, 2) + (-1)^{\gamma_{12}} \gamma_{12} \Omega^+(1, 1) \Omega^+(0, 1) \\ & + \Omega^+(1, 0) \left[\frac{1}{2} \gamma_{12} \Omega^+(0, 1) (\gamma_{12} \Omega^+(0, 1) + 1) \right] . \end{aligned} \quad (4.2)$$

The two contributions on the first line can be interpreted as the index of two-centered black hole solutions, carrying charges $\alpha_1 = \gamma_1$ and $\alpha_2 = 2\gamma_2$ for the first term, or $\alpha_1 = \gamma_1 + \gamma_2$ and $\alpha_2 = \gamma_2$ for the second term. Indeed, for such two-centered solutions, the distance is fixed to [12]

$$r_{12} = \frac{1}{2} \frac{\langle \alpha_1, \alpha_2 \rangle |Z(\alpha_1) + Z(\alpha_2)|}{\text{Im}[Z(\alpha_1)\bar{Z}(\alpha_2)]} , \quad (4.3)$$

which is positive on the side c_- of the wall only; the unit vector \vec{r}_{12}/r_{12} can be chosen arbitrarily on the unit two-sphere. Such configurations carry angular momentum $\vec{J} = \frac{1}{2}(\alpha_{12} - 1)\vec{r}_{12}/r_{12}$

(similar to the angular momentum carried by an electron in a magnetic monopole background), and therefore have $|\alpha_{12}|$ possible configurational states, with index $g(\alpha_1, \alpha_2) = (-1)^{\alpha_{12}} \alpha_{12}$. Since the distance r_{12} diverges in the vicinity of the wall, the internal degrees of freedom are decoupled from the configurational degrees of freedom, and the total index is the product $g(\alpha_1, \alpha_2) \Omega^-(\alpha_1) \Omega^-(\alpha_2)$, consistently with (4.2).

The contribution on the second line of (4.2) is more interesting. Letting $d = \gamma_{12} \Omega^+(0, 1)$, the term in bracket is recognized as the index of the symmetric (or, when $d < 0$, antisymmetric) part of the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_1$, where \mathcal{H}_1 is the space of quantum states accessible to one particle of charge γ_2 in the field of the particle of charge γ_1 . In other words, the second line corresponds to a configuration of two identical centers of charge γ_2 orbiting around a center of charge γ_1 , with Bose statistics when $d > 0$, or Fermi statistics when $d < 0$. More generally, (4.1) can be interpreted as the contribution from halos of particles of charge $k\gamma_2$ orbiting on a fixed shell around a center of charge $\gamma_1 + k_0\gamma_2$, and obeying Bose or Fermi statistics, depending on the sign of $\langle \gamma_1 + k_0\gamma_2, k\gamma_2 \rangle \Omega^+(k\gamma_2)$.

In contrast, in terms of the rational DT invariants the semi-primitive wall-crossing formula (2.22) reads

$$\frac{\sum_{N \geq 0} \Omega^-(1, N) q^N}{\sum_{N \geq 0} \Omega^+(1, N) q^N} = \exp \left(\sum_{\ell=1}^{\infty} (-1)^{\ell \langle \gamma_1, \gamma_2 \rangle} \ell \langle \gamma_1, \gamma_2 \rangle \bar{\Omega}(\ell\gamma_2, y) q^\ell \right), \quad (4.4)$$

so that, e.g. for $\gamma \mapsto \gamma_1 + 2\gamma_2$,

$$\Delta\Omega(1, 2) = 2\gamma_{12} \Omega^+(1, 0) \bar{\Omega}^+(0, 2) + (-1)^{\gamma_{12}} \gamma_{12} \Omega^+(1, 1) \Omega^+(0, 1) + \Omega^+(1, 0) \left[\frac{1}{2} (\gamma_{12} \Omega^+(0, 1))^2 \right], \quad (4.5)$$

where we combined $\Omega(0, 2)$ and $-\frac{1}{4}\bar{\Omega}(0, 1)$ in (4.2) into $\bar{\Omega}(0, 2)$. Unlike (4.2), the last term in (4.5) is of the form $\frac{1}{2}d^2$, which would be the result if the two particles of charge $(0, 1)$ were distinguishable and obeyed Boltzmann statistics. More generally, (4.4) can be interpreted as contributions of the same halo of particles with charge $k\gamma_2$ described above, but now satisfying Boltzmann statistics. While (4.5) is hardly shorter than (4.2), the reader can easily convince him/herself of the power of this simplification by computing $\Delta\Omega(1, N)$ for higher N .

The lesson to take from this discussion is that, rather than computing the variation of $\Omega(\gamma, t)$ across the wall $\mathcal{W}(\gamma_1, \gamma_2)$, it is advantageous to compute instead the variation of the rational invariants $\bar{\Omega}(\gamma, t)$ defined in (2.13), and apply the following recipe: treat the centers as distinguishable pointlike particles and compute their configurational index. Whenever m centers carry the same charge α , divide the configuration index by a Boltzmann-Gibbs factor $1/m!$. Finally, multiply the configurational index by the effective (rational) index $\bar{\Omega}(\alpha, t)$ carried by each of the centers. The result of this recipe is the formula (1.4), where $g(\{\alpha_i\})$ is identified as the configurational index of the quantum mechanics of n distinguishable particles interacting via Coulomb and Lorentz type forces (which we discuss in detail in the next subsection). Although we motivated this recipe by inspecting the classical semi-primitive formula, it in fact holds in full generality and applies to the refined (or motivic) index as well, see [1] for more details.

4.2. The phase space of multi-centered BPS black holes

Let us now review some relevant properties of supersymmetric multi-centered black hole solutions in $\mathcal{N} = 2$ supergravity (a similar analysis for multi-centered dyon solution in the low energy limit of $\mathcal{N} = 2$ gauge theories can be found in [30, 31]). Such solutions fall into the stationary metric ansatz

$$ds^2 = -e^{2U} (dt + \mathcal{A})^2 + e^{-2U} d\vec{r}^2 \quad (4.6)$$

where the scale function U , the Kaluza-Klein one-form \mathcal{A} and the vector multiplet scalars $z^a, a = 1 \dots n_v$ depend on the coordinate \vec{r} on \mathbb{R}^3 .

For n centers located at $\vec{r}_1, \dots, \vec{r}_n$, carrying electromagnetic charges $\alpha_1, \dots, \alpha_n \in \Gamma$ with total charge $\gamma = \alpha_1 + \dots + \alpha_n$, the values of the vector multiplet scalars t and of the scale factor U are obtained by solving [12].

$$-2 e^{-U(\vec{r})} \text{Im} \left[e^{-i\phi} Y(t(\vec{r})) \right] = \beta + \sum_{i=1}^n \frac{\alpha_i}{|\vec{r} - \vec{r}_i|}, \quad \phi = \arg Z(\gamma, t_\infty), \quad (4.7)$$

where $Y(t) = -e^{\mathcal{K}/2}(X^\Lambda(t), F_\Lambda(t))$ is the symplectic section afforded by the special geometry of the vector multiplet moduli space, such that $Z(\gamma, t) = \langle \gamma, Y(t_\infty) \rangle$. The constant vector β on the right-hand side of (4.7) is determined in terms of the asymptotic values of the moduli at infinity t_∞ by

$$\beta = -2 \text{Im} \left[e^{-i\phi} Y(t_\infty) \right]. \quad (4.8)$$

In particular, it follows from (4.7) that the scale factor U is given by evaluating the Bekenstein-Hawking entropy function $S(\gamma)$ on the harmonic function appearing on the right-hand side of (4.7) [32],

$$e^{-2U(\vec{r})} = \frac{1}{\pi} S \left(\beta + \sum_{i=1}^n \frac{\alpha_i}{|\vec{r} - \vec{r}_i|} \right). \quad (4.9)$$

Most importantly for our purposes, the locations \vec{r}_i are subject to the condition of mechanical equilibrium under the Coulomb, Lorentz, Newton and scalar exchange forces (also known as integrability equations) [12]

$$\sum_{\substack{j=1 \\ j \neq i}}^n \frac{\alpha_{ij}}{r_{ij}} = c_i, \quad (4.10)$$

where $r_{ij} = |\vec{r}_i - \vec{r}_j|$, $\alpha_{ij} \equiv \langle \alpha_i, \alpha_j \rangle$, and the real constants

$$c_i \equiv 2 \text{Im} [e^{-i\phi} Z(\alpha_i, t_\infty)] \quad (4.11)$$

depend on the asymptotic values of the moduli. Since $\phi = \arg Z(\gamma, t_\infty)$, the constants c_i satisfy $\sum_{i=1}^n c_i = 0$. The conditions (4.10) guarantee the existence of a Kaluza-Klein connection \mathcal{A} such that the above configuration is a supersymmetric solution of the equations of motion. In order for the solution to be physical however, one must also require that the scale factor be everywhere positive

$$S \left(\beta + \sum_{i=1}^n \frac{\alpha_i}{|\vec{r} - \vec{r}_i|} \right) > 0, \quad \forall \vec{r} \in \mathbb{R}^3, \quad (4.12)$$

where \vec{r}_i is the location of the i -th center. For the configurations relevant to the wall-crossing problem, this condition appears to be automatically satisfied.

Now, let $\mathcal{M}_n(\{\alpha_{ij}\}; \{c_i\})$ be the space of solutions $\{\vec{r}_1, \dots, \vec{r}_n\}$ to the equilibrium conditions (4.10), modulo overall translations of the centers. \mathcal{M}_n is a (possibly disconnected) $2n - 2$ -dimensional submanifold of $\mathbb{R}^{3n-3} \setminus \Delta$, where Δ is the locus in \mathbb{R}^{3n-3} where two or more of the centers \vec{r}_i coincide. $\mathbb{R}^{3n-3} \setminus \Delta$ is equipped with the closed two-form

$$\omega = \frac{1}{4} \sum_{i < j} \alpha_{ij} \frac{\epsilon^{abc} dr_{ij}^a \wedge dr_{ij}^b r_{ij}^c}{|r_{ij}|^3}. \quad (4.13)$$

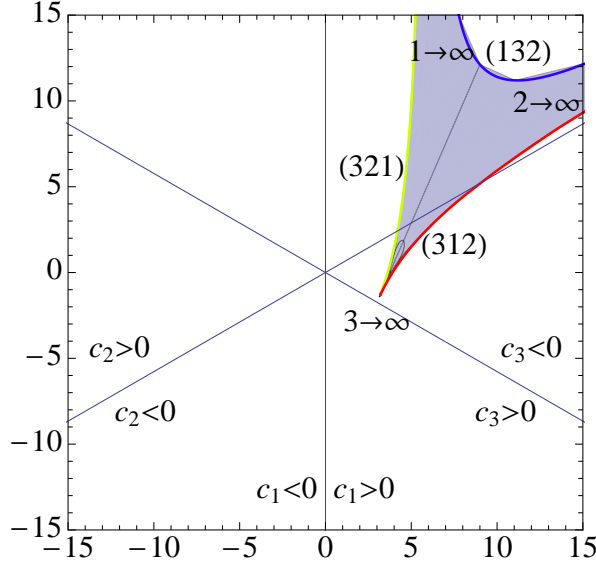


Figure 2. Phase structure of the moduli space \mathcal{M}_n of 3-centered solutions as a function of c_i , for fixed charges such that $\alpha_{12} > 0, \alpha_{23} > 0, \alpha_{13} > 0, \alpha_{12} < \alpha_{23}$. The shaded area represent the values of c_i in the two-dimensional section $c_1 + c_2 + c_3 = 0$ which are spanned as the location of the 3rd center is varied, keeping the centers 1 and 2 fixed. Conversely, if the values of c_i is fixed, the range of distances between the centers 1 and 2 can be read off by intersecting the shaded area with a radial line which joins c_i to the origin. Thus, for this choice of charges, 3-centered solutions only exist in the region $c_1 > 0, c_3 < 0$. Inside this region, the range of r_{12} is bounded from below and from above, except on the wall of marginal stability $c_2 = 0$. The boundaries of the shaded region correspond to collinear solutions whose order is indicated. As the wall is crossed, the topology of the collinear solutions changes from (321), (132) to (321), (312).

For generic values of c_i , the restriction of ω to \mathcal{M}_n is non-degenerate and endows \mathcal{M}_n with a symplectic structure [16]. Moreover, the symplectic form ω is invariant under $SO(3)$ rotations. The moment map associated to infinitesimal rotations is the angular momentum

$$\vec{J} = \frac{1}{2} \sum_{i < j} \alpha_{ij} \frac{\vec{r}_{ij}}{|r_{ij}|} . \quad (4.14)$$

Away from walls of marginal stability, the distances r_{ij} are bounded from above. If it is possible to order the α_i 's such that $\langle \alpha_i, \alpha_j \rangle \geq 0$ whenever $i \leq j$, as it is the case when the α_i 's lie in a two-dimensional cone $\tilde{\Gamma}$, the distances r_{ij} are also bounded from below by a non-zero $r_{\min} > 0$, and the space \mathcal{M}_n is therefore compact. E.g. for two centers, $\mathcal{M}_2 = S^2$ equipped with $\omega = \frac{1}{2} \alpha_{12} \sin \theta d\theta d\phi$ for $\text{sign}(\alpha_{12}) = \text{sign}(c_1)$, and zero otherwise. A representative of the phase structure of \mathcal{M}_3 is illustrated in Figure 2.

4.3. Equivariant index and localization

The symplectic space $\mathcal{M}_n \equiv \mathcal{M}_n(\{\alpha_{ij}\}; \{c_i\})$ defines the classical phase space of the configurational degrees of freedom of n -centered BPS black hole solutions. Since such configurations are stationary, the Hamiltonian vanishes and all points in \mathcal{M}_n are degenerate in energy. Quantum mechanically, the Hilbert space consists of sections of $\mathcal{H} = S \otimes \mathcal{L}$, where

$S = S_+ \oplus S_-$ is the total spin bundle⁴ over \mathcal{M}_n and \mathcal{L} is a complex line bundle over \mathcal{M}_n with first Chern class ω . BPS states correspond to zero-modes of the Dirac operator D on \mathcal{M}_n , and are analogous to states in the lowest Landau level for an electron immersed in a magnetic flux ω . The Dirac operator decomposes as $D = D_+ + D_-$ where D_+ maps $S_+ \otimes \mathcal{L}$ to $S_- \otimes \mathcal{L}$ and vice-versa. The action of $SO(3)$ on \mathcal{M}_n lifts to an action of $SU(2)$ on $S_{\pm} \otimes \mathcal{L}$, and the refined index is then

$$g_{\text{ref}}(\{\alpha_i\}, y) = \text{Tr}_{\text{Ker}D_+}(-y)^{2J_3} + \text{Tr}_{\text{Ker}D_-}(-y)^{2J_3} \quad (4.15)$$

where J_3 is the operator representing the rotations along the z axis. Assuming that $\text{Ker}D_- = 0$, it follows that the refined index is equal to the equivariant index of the Dirac operator D ,

$$g_{\text{ref}}(\{\alpha_i\}, y) = \text{Tr}_{\text{Ker}D_+}(-y)^{2J_3} - \text{Tr}_{\text{Ker}D_-}(-y)^{2J_3} \quad (4.16)$$

The assumption that $\text{Ker}D_- = 0$ can be proven when \mathcal{M}_n is Kähler (which is the case when $n = 2, 3$). We do not know how to prove it in general, but it is supported by the fact that it leads to results in agreement with the KS and JS formulae. In §6, we speculate that this assumption may be unnecessary if one were to compute the jump of the protected spin character in the context of $\mathcal{N} = 2$ SYM theories.

Now, by the Atiyah-Bott Lefschetz fixed point formula [33, 34, 35, 36], the equivariant index localizes to the fixed points of the action of J_3 on \mathcal{M}_n . Clearly, those correspond to solutions where all centers lie along the z axis, and satisfy the one-dimensional equilibrium conditions

$$\sum_{\substack{j=1 \\ j \neq i}}^n \frac{\alpha_{ij}}{|z_j - z_i|} = c_i, \quad \sum_{i=1}^n z_i = 0, \quad (4.17)$$

where the last equation fixes the translational zero-mode. For any $\sigma \in \mathcal{S}_n$, where \mathcal{S}_n denotes the set of permutations of $\{1, \dots, n\}$, we denote by $\mathcal{C}(\sigma)$ the set of solutions to (4.17) such that $z_{\sigma(i)} < z_{\sigma(j)}$ if $i < j$. The set $\mathcal{C}(\sigma)$ corresponds to the subset of the critical points of the ‘superpotential’

$$W(\lambda, \{z_i\}) = - \sum_{i < j} \alpha_{ij} \text{sign}(z_j - z_i) \ln |z_j - z_i| - \sum_i (c_i - \lambda/n) z_i \quad (4.18)$$

which are ordered according to the permutation σ . In the vicinity of a fixed point $p \in \mathcal{C}(\sigma)$, the angular momentum J_3 and the symplectic form ω take the form

$$J_3 = \frac{1}{2} \sum_{i < j} \alpha_{\sigma(i)\sigma(j)} - \frac{1}{4} M_{ij}(p) (x_i x_j + y_i y_j) + \dots, \quad \omega = \frac{1}{2} M_{ij}(p) dx_i \wedge dy_j + \dots, \quad (4.19)$$

where M_{ij} is the Hessian matrix of $W(\lambda, \{z_i\})$ with respect to z_1, \dots, z_n , and (x_i, y_i) are coordinates in the plane transverse to the z -axis at the center i , subject to the condition $\sum_i x_i = \sum_i y_i = 0$. Except for an overall translational zero-mode, the matrix M_{ij} is non-degenerate, and the critical points are isolated, so $\mathcal{C}(\sigma)$ is a finite set (possibly empty).

The Lefschetz fixed point formula of [35] yields an explicit formula for the refined index

$$g_{\text{ref}}(\{\alpha_i\}, y) = \frac{(-1)^{\sum_{i < j} \alpha_{ij} + n - 1}}{(y - 1/y)^{n-1}} \sum_{\sigma \in \mathcal{S}_n} s(\sigma) y^{\sum_{i < j} \alpha_{\sigma(i)\sigma(j)}}, \quad (4.20)$$

⁴ We assume that \mathcal{M}_n is spin, see [11] for a discussion of this issue.

where $s(\sigma)$ counts (with sign) the number of solutions to (4.17) ordered according to the permutation σ ,

$$s(\sigma) = - \sum_{p \in \mathcal{C}(\sigma)} \text{sign} \det \hat{M}, \quad (4.21)$$

where \hat{M} is the Hessian of W as a function of the $n+1$ variables λ, z_1, \dots, z_n , evaluated at the given solution of (4.17). The factor $(y - 1/y)^{n-1}$ in (4.20) originates from the equivariant \hat{A} -genus in the Lefschetz fixed point formula. It is convenient to let the sum in (4.20) run over all permutations and set $s(\sigma) = 0$ when there are no solutions to (4.17) in the order specified by σ . For reasons that will become clear in §5, we refer to (4.20) as the ‘Coulomb branch wall-crossing formula’.

The formula (4.20) is fully explicit, yet it depends on our ability to find solutions of the one-dimensional problem (4.17). While this can be done numerically (approximate solutions are sufficient since the answer depends only on the order σ and the sign of $\det M$), it would be useful to have a general criterium for determining when solutions exist, and if so to compute their Morse index. The answer to these questions is suggested by an alternative approach based on quivers (see §6).

At this point, we can check whether (4.20) agrees with the answer of the KS or JS formulae. For $n = 2$, we find two fixed points with permutations $\sigma(12) = 12$ and 21 , leading to

$$g(\alpha_1, \alpha_2; y) = (-1)^{\alpha_{12}} \frac{\sinh(\nu \alpha_{12})}{\sinh \nu}, \quad (4.22)$$

in agreement with (2.19). For $n = 3$ (and for the same choice of α_i as above (2.20)), we find 4 possible orderings, with the following value of $s(\sigma)$,

$$\{1, 2, 3; +\}, \{2, 1, 3; -\}, \{3, 1, 2; -\}, \{3, 2, 1; +\}, \quad (4.23)$$

leading to

$$\begin{aligned} g_{\text{ref}}(\alpha_1, \alpha_2, \alpha_3, y) &= (-1)^{\alpha_{12} + \alpha_{23} + \alpha_{13}} (y - y^{-1})^{-2} \\ &\times \left(y^{\alpha_{12} + \alpha_{13} + \alpha_{23}} - y^{\alpha_{13} + \alpha_{23} - \alpha_{12}} - y^{\alpha_{12} - \alpha_{23} - \alpha_{13}} + y^{-\alpha_{12} - \alpha_{13} - \alpha_{23}} \right), \end{aligned} \quad (4.24)$$

in agreement with the result (2.20). Further checks for $n > 3$ can be found in [1]. It is an open problem to show by combinatorial means that the result (4.20) agrees with the KS formula in general.

5. Wall-crossing from Abelian quivers

Finally, we discuss an alternative computation of the combinatorial factors $g_{\text{ref}}(\{\alpha_i\}, y)$ based on Reineke’s formula for invariants of quivers without closed loop.

The physical idea is that the supersymmetric quantum mechanics of multi-centered black holes admits two branches: the Coulomb branch, where the centers are far separated and well described by the supergravity solution described in §4.2, and the Higgs branch, where the centers are very close to each other and are better represented as D-branes, with open strings stretched between them. At large string coupling, the wave function is mainly supported on the Coulomb branch, while at small string coupling it is mainly supported on the Higgs branch [13]. However, the BPS index $\text{Tr}'(-1)^F$ is independent of the string coupling (and other hypermultiplet fields), and should be computable in both. It is less clear that the same should be true of the refined index $\text{Tr}'(-y)^{2J_3}$, since this quantity is in general non-protected in string theory [28], but for

what concerns the jump of the refined index across walls of marginal stability, we shall find strong evidence that this is the case.

On the Higgs branch, the D-brane system is described at low energy by a supersymmetric quiver quantum mechanics, with gauge group $\prod_{i=1}^n U(N_i)$, where N_i is the number of coinciding D-branes at point i , and $\langle \alpha_i, \alpha_j \rangle$ fields in the bifundamental representation (N_i, \bar{N}_j) when $\langle \alpha_i, \alpha_j \rangle$ is positive, or $-\langle \alpha_i, \alpha_j \rangle$ fields in the bifundamental representation (\bar{N}_i, N_j) in the opposite case. Thanks to the Bose-Fermi/Boltzmann correspondence described in §4.1, we can treat all centers as distinguishable and assume that $N_i = 1$ for any i , provided that we attach an effective rational index $\bar{\Omega}_{\text{ref}}(\alpha_i, y)$ to each center, and perturb the charge vectors so that none of them coincide. Moreover, since the α_i can be ordered such that $\langle \alpha_i, \alpha_j \rangle \geq 0$ whenever $i < j$, the quiver admits no closed loop.

For arbitrary quivers without loops, Reineke has computed the Poincaré polynomial of the moduli space of quiver representations [37]. In the special case of Abelian quivers ($N_i = 1$), Reineke's formula gives

$$g_{\text{ref}}(\{\alpha_i\}, y) = \frac{(-y)^{-\sum_{i < j} \alpha_{ij}}}{(y - 1/y)^{n-1}} \sum_{\text{partitions}} (-1)^{s-1} y^{2\sum_{a \leq b} \sum_{j < i} \alpha_{ji} m_i^{(a)} m_j^{(b)}}, \quad (5.1)$$

where the sum runs over all ordered partitions of $\gamma = \alpha_1 + \dots + \alpha_n$ into s vectors $\beta^{(a)}$ ($1 \leq a \leq s$, $1 \leq s \leq n$) such that

- (i) $\beta^{(a)} = \sum_i m_i^{(a)} \alpha_i$ with $m_i^{(a)} \in \{0, 1\}$, $\sum_a \beta^{(a)} = \gamma$
- (ii) $\langle \sum_{a=1}^b \beta^{(a)}, \gamma \rangle > 0 \quad \forall \quad b \quad \text{with} \quad 1 \leq b \leq s-1$

We refer to (5.1) as the ‘Higgs branch wall-crossing formula’.

Let us illustrate how this formula works by computing $g_{\text{ref}}(\{\alpha_i\}, y)$ for $n = 2, 3$. For $n = 2$ case with $\alpha_{12} > 0$, there are two possible ordered partitions satisfying the conditions stated above:

$$\{\alpha_1 + \alpha_2\}, \quad \{\alpha_1, \alpha_2\}. \quad (5.2)$$

The first term contributes $y^{2\alpha_{12}}$ and the second term contributes -1 to the sum. In total,

$$g_{\text{ref}}(\alpha_1, \alpha_2, y) = (-y)^{1-\alpha_{12}} (y^2 - 1)^{-1} (y^{2\alpha_{12}} - 1) = (-1)^{\alpha_{12}+1} \frac{y^{\alpha_{12}} - y^{-\alpha_{12}}}{y - y^{-1}}, \quad (5.3)$$

in agreement with (2.19). For $n = 3$, assuming the same conditions on α_i as in (2.20), we find 6 possible ordered partitions

$$\{\alpha_1 + \alpha_2 + \alpha_3\}, \quad \{\alpha_1, \alpha_2 + \alpha_3\}, \quad \{\alpha_1 + \alpha_2, \alpha_3\}, \quad \{\alpha_1 + \alpha_3, \alpha_2\}, \quad \{\alpha_1, \alpha_2, \alpha_3\}, \quad \{\alpha_1, \alpha_3, \alpha_2\}. \quad (5.4)$$

The second and the last contribution cancel, leaving

$$g_{\text{ref}}(\alpha_1, \alpha_2, \alpha_3, y) = (-1)^{\alpha_{12}+\alpha_{13}+\alpha_{23}} (y - y^{-1})^{-2} \left(y^{\alpha_{12}+\alpha_{13}+\alpha_{23}} - y^{\alpha_{12}-\alpha_{23}-\alpha_{13}} - y^{\alpha_{13}+\alpha_{23}-\alpha_{12}} + y^{-\alpha_{12}-\alpha_{13}-\alpha_{23}} \right), \quad (5.5)$$

in agreement with (2.20).

6. Conclusion and open problems

In this survey, we have described four apparently different wall-crossing formulae, the KS formula (2.9), the JS formula (3.6), the ‘Coulomb branch’ formula (4.20) and the ‘Higgs branch’ formula (5.1). In [1] we have checked that these formulae agree among each other for $n \leq 5$.

Unfortunately, we do not have a mathematical proof of their equivalence at this point. The equivalence of the KS and JS formula appears to be on solid ground [10, 7]. The Reineke formula (or rather, its specialization at $y = 1$) is known to follow from the JS formula [38] (although I do not know of an explicit combinatorial proof). The underlying mathematical structure of these formulae involves Ringel-Hall algebras (and generalizations thereof [39]), which might provide a realization of the long sought-after algebra of BPS states [40, 41].

The Coulomb branch formula (4.20) appears to be new, and it would be desirable to derive it e.g. from the Higgs branch formula (5.1). Indeed, the equivalence between these two formulae could be viewed as a toy model of open/closed string duality. The comparison of the Higgs and Coulomb branch formulae suggests that the permutations σ for which $s(\sigma)$ does not vanish are those whose maximal increasing subsequence⁵ satisfies the condition ii) below (5.1), while none of its non-maximal increasing subsequence does, in which case $s(\sigma) = (-1)^{\#\{i; \sigma(i+1) < \sigma(i)\}}$. It would be interesting to derive these conditions by applying Morse theory to the ‘superpotential’ W in (4.18).

While the Coulomb and Higgs branch formulae have been derived in the context of $\mathcal{N} = 2$ supergravity, the fact that they appear to agree with the KS and JS formulae suggests that they should work just as well in the context of $\mathcal{N} = 2$ SYM theories. For such field theories, unlike in supergravity, the refined index (or rather a variant thereof, known as the protected spin character, and constructed out of the spatial rotation generator J_3 and a $SU(2)_R$ symmetry generator I_3 [28]) receives contributions from short multiplets only. It would be desirable to derive the jump of the protected spin character by quantizing multi-centered dyonic solutions of the low energy Abelian gauge theory, along the lines of [30, 31], and see if it agrees with the Coulomb formula (presumably this jump will coincide with the equivariant index of the Dirac operator D , without the need to assume that $\text{Ker } D_- = 0$). It would also be interesting to clarify the relation with other realizations of BPS dyons in $\mathcal{N} = 2$ gauge theories based on string webs or non-Abelian monopoles [42, 43, 44, 45, 46].

More generally, localization techniques appear to be a powerful way of quantizing multi-centered black hole solutions at fixed values of the moduli, not only in the vicinity of walls of marginal stability. In contrast to the situation studied here, the phase space \mathcal{M}_n is in general no longer compact, due to the presence of scaling solutions [13, 47, 48, 14, 49]. We refer the reader to [11] for an application of localization techniques to this problem.

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⁵ Rather than defining the notion of (maximal) increasing subsequence, we illustrate it on an example: for the permutation $\sigma(1234) = 3142$, the increasing subsequences are $\{\{3\}, \{14\}, \{2\}\}$ and $\{\{3\}, \{1\}, \{4\}, \{2\}\}$, associated to the ordered decompositions $\{\alpha_2, \alpha_1 + \alpha_4, \alpha_3\}$ and $\{\alpha_2, \alpha_4, \alpha_1, \alpha_3\}$, respectively. The first increasing subsequence is maximal, the second is not. More details can be found in [1], §3.3.

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